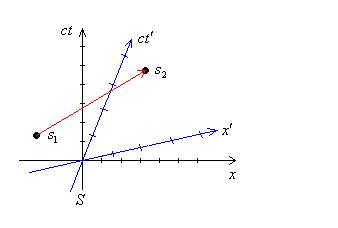
**Special Relativity Geometry**

So now I want to introduce the concept of distance, the metric, basis vectors, and discuss scalars, tensors, etc. First up, the invariant interval, which is necessary to have before we can write down a metric, etc., etc.

**1. The invariant interval**

Consider two events happening somewhere in space-time,



Let’s calculate the ‘distance’ between these two events in both the S and S′ reference frame. The concept of distance between space-time points is a little ambiguous. It is somewhat arbitrary how one defines such a quantity. Certainly one could say



But it turns out this is not the way we want to do it because this definition will not result in a quantity which is invariant w/r to all frames of reference – meaning that if we compute this interval in reference frame S, we will not find it to be the same magnitude if we compute it in S′. But the distance between two events, in a space-time sense, ought to be constant regardless of the coordinates one uses to express it in. In a similar way, the distance between two points in the Cartesian x-y plane is the same whether we express it in the x-y coordinates, or in a set of u-v coordinates rotated w/r to the x-y by some angle. So this is what rules out this quantity as being a measure of distance. But there is a related quantity which is invariant.



And so we take this to be the distance between events. So in general for a particle traveling through space, Δs2 ≤ 0. And the closer it gets to 0, the faster it’s going. Only light (and other sorts of waves) reaches this speed. Let’s prove Δs2 is invariant.

**Example**

Show that (cΔt)2 +(Δx)2 isn’t invariant w/r to Lorentz transformation, while –(cΔt)2 + (Δx)2 is.

So we form this quantity in the reference S′ traveling to the right at speed v w/r to S.



putting this in terms of S coordinates via the Lorentz transformation



we have:



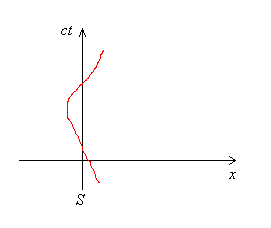
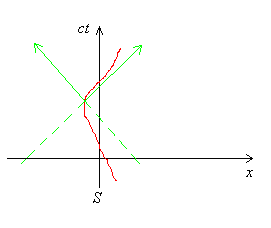
But now let’s look at the other possibility, (Δs)2 = -(cΔt)2 + (Δx)2. So writing out this distance in terms of the S′ coordinates, and then putting in terms of the S coordinates we have:



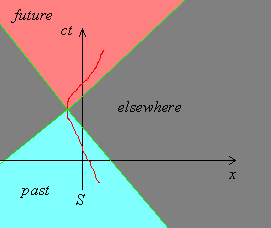
So it appears that *this* is the invariant interval – the true representation of the ‘distance’ between space time events. This is our first, and conclusive, indication that the geometry of space-time is *not* Euclidean.

**Time-like, space-like, and null intervals**

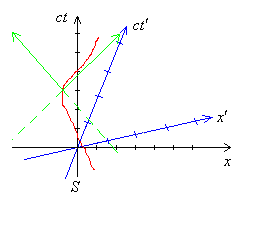
Consider a particle traveling along the x axis according to the equation x(t). Then its path can be represented in the S reference frame as:

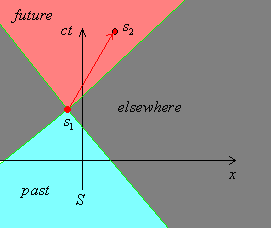
Note that at any given point in time, the path of the particle is limited to the so-called light-cone, illustrated in green. The light cone is the path taken by pulse of light (it goes in positive and negative directions simultaneously) emitted at a particular point in space time. The angle the light cone makes with the axes is 45° since we should have slope = ct/x = c(t/x) = c/c = 1. The path can never cross the light cone since that would imply faster than light travel. The light cone separates space-time into 3 regions: past, future, and elsewhere.



The past is the region of space/time that it could’ve been previously. The future is where it could go. And elsewhere is where it can never be – because that would imply faster than light travel. We can also look at this motion from the perspective of S′. It would look like,



Notice that the light cone for the S′ reference frame is the same as that for the S reference frame because in the S′ reference frame, the measured speed of light must also be c = x′/t′ → 1 = x′/ct′ → slope of light cone = 1. And so the past/future/elsewhere regions are also the same. Thus special relativity doesn’t allow the possibility of going to a frame of reference where the future and past of a particle can be reversed – thus preserving causality (sorry). Let’s look again at two events in the S reference frame.



Suppose we draw the light cone originating from s1. Then if it encompasses s2 (i.e. if s2 is in the future or past), then the distance between these two events is called time-like. If s2 is on the light cone itself, then it is called ‘null’. And if s2 is in the elsewhere region it is called space-like. Observe that in terms of value of Δs2 itself, we could say equivalently,



because for instance, if s2 occurs within the future of s1, then the speed required to go from s1 to s2 must be less than that of light. And so we must have v < c → v2 < c2 →

-c2Δt2 + v2Δt2 < 0 → -c2Δt2 + Δx2 < 0 → Δs2 < 0. And similarly for the other two cases.

**2. The metric and basis vectors**

Anyway, we would like to proceed and write down the metric for our space-time. It is (implicit summation over repeated indices):



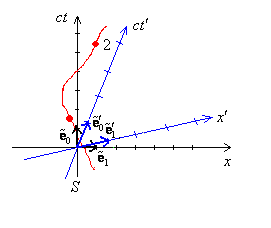
where,



[since (cdt)2 – dx2 – dy2 – dz2 is also invariant, we could have chosen this as the invariant interval and accordingly defined ηαβ as diag(1,-1,-1,-1), which is a choice made in Particle Physics] The negative sign makes all the difference between Euclidean and relativistic space-time. Alright, now we would like to endow our space-time with unit vectors pointing along the coordinate axes. These are called the covariant basis vectors, as mentioned previously in the tensor notes. And we will also like to introduce the contravariant basis vectors as well, which are orthonormal to the covariant basis.



The covariant basis vectors point along the ct, x, y, and z axes just as in Cartesian coordinates, illustrated below for the S and S′ axes,



And the normalization (inner product) of the covariant basis follows from the metric. We have:



and so,



So you’ll observe that the time-like basis vector has magnitude (-1), rather unlike we’re used to in Cartesian coordinates. So this metric certainly doesn’t correspond to our usual conception of the ‘dot’ produce **A·B** = ABcosθ. This is how space-time differs from normal Euclidean space. Last, we’ll see that the metric serves as an ‘index raising’ operator. We can expand the covariant basis in terms of the contravariant one. For instance,



So we can say:



And reversing the process, we can say:



But dotting both sides by **e**γ we have:



Defining the LHS as ηαγ, we have,



And so altogether,



**3. How metric and basis vectors transform under Lorentz transformation**

So we can see that our basis vectors change when we go to different reference frames. First we’ll consider the metric. So we know the coordinates in the S´ frame relate to those of the S frame via the matrix equation (implicit summation, and note that in Tensors file, I don’t put primes on indices of primed system, instead I use greek indices to represent primed system, and latin indices to represent unprimed system):



and therefore how the metric changes in general,



so that:



We can verify that this is the same as ηαβ itself (see Tensors file Appendix on how highlighted equation can be converted to matrix multiplication).



as it should be for Lorentz transformations. Anyway, this also tells us how the basis vectors change, because:



which clearly suggests:



A similar analysis on ηαβ will show that:



So altogether,



**4. Scalars, Vectors, Tensors**

Scalars are quantities which do not change their value when going from one coordinate system to another. Vectors and tensors are similar. They also do not change their value, but their value includes both magnitude and direction.

**Definition of tensors in terms of basis vectors**

We can write a space-time vector (also called a 4-vector) as, for instance:



where Tα are called the contravariant components of the vector . We can also write them in the contravariant basis as:



where Tα are called the covariant components of the vector. Along the same lines, we can write a space-time tensor as:



where Tαβ are called the contravariant components of the tensor . Or we can write it as:



where Tαβ are called the mixed components of the tensor . And could write as:



where Tαβ constitute a different set of mixed Tensor components. And finally we can write it as:



where Tαβ are called the covariant components of the tensor . And we could generalize to even higher order tensors but there is usually no need to here. There is a relationship between the contravariant and covariant components. We have:



So we have the relation between the upper index components and lower index components of a vector. Similarly we can find that:



We also have:



Where we define ηαβ as the inner product of **e**α and **e**β, which is actually δαβ. So summarizing our results we have:



So we see that the metric also functions as a ‘raising/lowering’ operator. This is true for more complicated elements, like tensor components, so that:



etc.

**Effects of Lorentz transformation on tensor components**

So generally, the components of a tennsor change according to:



They must transform this way in order to keep space-time vectors invariant under a Lorentz transformation – which is the defining characteristic of a vector. For instance we’ll have:



and so we see that the vector remains the same in all bases. And it so follows that tensor components transform as:



You can work this out yourself if you wish. It follows from the transformation laws of the basis vectors and the fact that a tensor must be invariant w/r to Lorentz transformations.

**5. Testing whether some construction is in fact a scalar/vector/tensor**

Now that we know how vector components transform, and basis vectors transform, we can determine whether or not quantities constructed in terms of these basis vectors are in fact, ‘vectors’. For instance, consider some set of quantities T(α) that transform under some rule when coordinates are changed. In order for T(α)to constitute the components of a vector, its transformation rule must be identical to the transformation rules of Tαor Tα. According to the first law of special relativity, the laws of physics must be covariant (nothing to do with covariant vector components above), i.e., written the same in any inertial frame, because no inertial frame is sacred. And implicit in this is that physical quantities are constructed covariantly (not to say they have the same functional form, though it so happens that constructed distance does, at least in SR, though not in GR, but that they are constructed in the same manner from physical observables) in each reference frame. This usually tells us how a quantity, such as T(α) is constructed in any given reference frame. The next question is: do T(α) constitute the components of some vector. To know this, we have to see if the covariant transformation is identical to the Tα or Tα transformation rule. If it matches up with the first, then T(α) form the contravariant components of a vector. If it matches up with the second, then T(α) form the covariant components of a vector. If neither, then T(α) do not form the components of a vector. Analogous arguments can be made for higher order quantities like T(α,β). If the transformation rule for T(α,β) matches the rule for Tαβ, Tαβ, Tαβ or Tαβ, then it can be said to form the mixed, contravariant, or covariant components of a tensor, . The reason we care if things can be considered as the components of vectors/tensors is because vector/tensor are things which exist ‘outside’ the reference frame, meaning that they are not changed when the reference frame is changed. As such, vector/tensor equations are not changed when the reference frame is changed. Another reason is that if an entity is known to be a tensor, then scalars can be constructed from it. For instance, the magnitude of any vector would be invariant in any reference frame too.

**Example**

Determine if the following quantities, which are defined to transform via same functional form, constitute the components of a vector.



Well we must examine the relation between Q(α) and Q´(α′). So changing variables and writing in a new coordinate system we have:



and so Q(α) does transform like a vector – the covariant components of a vector. This vector is:



which could be called the space-time gradient I suppose.

**Example**

Do the set of quantities below form the components of a vector, assuming they transform covariantly?



Well let’s see, forming Q(α) in another coordinate system,



so these quantities do not form the components of a vector.

**Example**

Let’s take a look again at the invariant interval-the distance between space-time events. In one frame it is:



And in another frame it is:



and we see that it is the same. In fact, the magnitude of any vector is going to be invariant w/r to Lorentz transformations. For example, consider a vector: **T** = Tα**e**α. Its magnitude is:



And in the S′ reference frame it would be:



which is the same as determined in the S reference frame. We’ll return to some of these ideas later when we discuss the generalization of Newton’s laws in SR.

**Example**

Suppose we have two events occurring at space-time points (ct,x) = (3,4) and (ct,x) = (11,-8) respectively in reference frame S. What are the coordinates of the event in a reference traveling with speed v = 0.4c to the left? What is the space-time interval in S and S′?

The coordinates in the reference frame going to the left are:



and,



The interval between events in S is:



The interval between the points in S′ is:



So there you go.